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On multivariate projection operators

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Abstract

This paper deals with multivariate Fourier series considering triangular type partial sums. Among others we give the exact order of the corresponding operator norm. Moreover, a generalization of the so-called Faber–Marcinkiewicz–Berman theorem has been proved.

Keywords: Rectangular partial sums; Triangular partial sums; Operator norm of the multivariate Fourier series; Projection operators

1. Introduction

1.1

Multivariate Fourier series has been the object of an intensive study. We may refer the classical works Zygmund [1, Ch. XVII] and Stein, Weiss [2, Ch. VII]. This paper is another contribution to this subject. To step further we need some notations.

Let $\mathbb{R}^d$ (direct product) be the Euclidean d-dimensional space ($d \geq 1$, fixed) and let $\mathbb{T}^d = \mathbb{R}^d \pmod{2\pi\mathbb{Z}^d}$ denote the d-dimensional torus, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Further, let $C(\mathbb{T}^d)$ denote the space of (complex valued) continuous functions on $\mathbb{T}^d$. By definition they are 2π-periodic in each variable.

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For $g \in C(\mathbb{T}^d)$ we define its Fourier series by

$$g(\boldsymbol{\vartheta}) \sim \sum_{\mathbf{k}} \hat{g}(\mathbf{k})\mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{\vartheta}}, \qquad \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\mathbf{t})\mathrm{e}^{-\mathbf{i}\mathbf{k}\cdot\mathbf{t}}\mathrm{d}\mathbf{t}, \tag{1.1}$$

where in the above vector notation $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_d) \in \mathbb{T}^d$, $\mathbf{k} = (k_1, k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $\mathbf{k}\cdot\boldsymbol{\vartheta} = \sum_{l=1}^d k_l \vartheta_l$ (scalar product).

The *rectangular* nth partial sum of the Fourier series is defined by

$$S_{nd}^{[r]}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_\infty \le n} \hat{g}(\mathbf{k})\mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{\vartheta}} \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}); \tag{1.2}$$

the *triangular* one is

$$S_{nd}^{[t]}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_1 \le n} \hat{g}(\mathbf{k})\mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{\vartheta}} \quad (n \in \mathbb{N}_0). \tag{1.3}$$

Above, $|\mathbf{k}|_\infty = \max_{1\le l\le d}|k_l|$ and $|\mathbf{k}|_1 = \sum_{k=1}^d |k_l|$ (they are the l_p norms of the multiindex $\mathbf{k}$). The names "rectangular" and "triangular" refer to the shape of the corresponding indices of terms when $d = 2$ and $0 \le k_1, k_2$, $|\mathbf{k}|_\infty \le n$, $|\mathbf{k}|_1 \le n$ respectively.

In a way the investigation of the $S_{nd}^{[r]}$ is apparent: in many cases in essence it is a one variable problem (see the above works [1,2]).

However there are only relatively few works dealing with the triangular (or l_1) summability: see Herriot [3].

In a recent paper [4] Berens and Xu prove the analogue of the famous Fejér paper on the $(C, 1)$ mean of $S_n(f)$ (here and later S_n stands for $S_{n1} \equiv S_{n1}^{[r]} \equiv S_{n1}^{[t]}$): actually it turns out that $(C, 2d-1)$ in a way corresponds to $(C, 1)$ for any $d \ge 1$. Moreover the explicit formula for the corresponding Dirichlet kernel D_{nd} is crucial: it gives the possibility to prove many theorem for $S_{nd} \equiv S_{nd}^{[t]}$ analogous to statements on S_n (this formula which actually gives D_{nd} as a divided difference of univariable function was developed earlier by Xu [5]).

Exploiting this relation we prove some theorems on S_{nd}; as it turns out they can be used to investigate $S_{nd}^{[r]}$, too.

2. New results

2.1

Introducing the notations

$$D_{nd}(\boldsymbol{\vartheta}) = \sum_{|\mathbf{k}|_1 \le n} \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{\vartheta}} \quad (n \ge 1), \tag{2.1}$$

where $\mathbf{k} \in \mathbb{Z}^d$, one can see that

$$\begin{aligned} S_{nd}(g, \boldsymbol{\vartheta}) = (g * D_{nd})(\boldsymbol{\vartheta}) &:= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\boldsymbol{\vartheta} - \mathbf{t})D_{nd}(\mathbf{t})\mathrm{d}\mathbf{t} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\boldsymbol{\vartheta} + \mathbf{t})D_{nd}(\mathbf{t})\mathrm{d}\mathbf{t}, \end{aligned} \tag{2.2}$$

where as before, $g \in C(\mathbb{T}^d)$, $\boldsymbol{\vartheta}, \mathbf{t} \in \mathbb{T}^d$ (cf. (1.3) or [2, Chs I, VII], [4]).

Let $\|g\| := \max_{\boldsymbol{\vartheta}\in\mathbb{T}^d} |g(\boldsymbol{\vartheta})|$,

$$\|S_{nd}\| := \max_{\substack{g\in C(\mathbb{T}^d)\\ \|g\|\le 1}} \|S_{nd}(g,\boldsymbol{\vartheta})\| \quad (n \ge 1)$$

and

$$\|g\|_p := \left(\int_{\mathbb{T}^d} |g(\boldsymbol{\vartheta})|^p \mathrm{d}\boldsymbol{\vartheta}\right)^{1/p}$$

if $g \in L^p :=$ {the set of all measurable 2π periodic (in each variable) functions on $\mathbb{T}^d$}, $1 \le p < \infty$.

We state

Theorem 2.1. *We have, for any fixed $d \ge 1$,*

$$\|D_{nd}\|_1 = \|S_{nd}\| \sim (\log n)^d \quad (n \ge 2).^{1} \tag{2.3}$$

Remark. The general case of the upper estimation (when $d \ge 4$) is due to Professor Gábor Halász [6]. We shall give another argument applying a formula of Xu.

The lower estimation also belongs to Gábor Halász [6]. The proof is based on his original argument; we thank him for communicating the ideas.

2.2

One of the most characteristic properties of the Fourier series in one dimension is the so-called Faber–Marcinkiewicz–Berman theorem, namely that the operator S_n has the smallest norm among all projection operators (cf. [7, p. 281] for other details). This part extends the above statement for S_{nd}, $d \ge 1$.

Let $\mathcal{T}_{nd}$ be the space of trigonometric polynomials of form

$$\sum_{|\mathbf{k}|_1\le n} \left(a_{\mathbf{k}} \cos(\mathbf{k}\cdot\boldsymbol{\vartheta}) + b_{\mathbf{k}} \sin(\mathbf{k}\cdot\boldsymbol{\vartheta})\right),$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_d)$ and $k_1, \ldots, k_d \ge 0$, arbitrary real numbers. Moreover, let T_{nd} be a linear trigonometric projection operator on $C(\mathbb{T}^d)$, i.e. $T_{nd}(g,\boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta})$ for $g \in \mathcal{T}_{nd}$ and $T_{nd}(g,\boldsymbol{\vartheta}) \in \mathcal{T}_{nd}$ for other $g \in C(\mathbb{T}^d)$.

Theorem 2.2. *For any linear trigonometric projection operator T_{nd}, one has*

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t})\mathrm{d}\mathbf{t} = S_{nd}(g,\boldsymbol{\vartheta}) \quad (g \in C(\mathbb{T}^d)), \tag{2.4}$$

$$\|T_{nd}\| \ge \|S_{nd}\|, \tag{2.5}$$

where $g_{\mathbf{t}}(\boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta} + \mathbf{t})$ is the $\mathbf{t}$-translation operator.

[1] Here and later $a_n \sim b_n$ means that $0 < c_1 \le a_n b_n^{-1} \le c_2$ where $c, c_1, c_2, \ldots$ are positive constants, not depending on n; they may denote different values in different formulae.

2.3

Here we formulate an analogue of Theorem 2.2 which can be applied for the norm estimation of projection onto the space Π_{nd} (the algebraic analogue of $\mathcal{T}_{nd}$; cf. [7, p. 284]). Let T_{nd} be a linear trigonometric projection operator defined on the even functions in $C(\mathbb{T}^d)$. We extend it to all of $C(\mathbb{T}^d)$ by defining $T^*_{nd}(g) := T_{nd}(G)$, where $2G(\boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta}) + g(-\boldsymbol{\vartheta})$. Obviously, $\|T^*_{nd}\| = \|T_{nd}\|$.

Theorem 2.3. *We have for $g \in C(\mathbb{T}^d)$*

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} T^*_{nd}(g_{\mathbf{t}} + g_{-\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t})\mathrm{d}\mathbf{t} = S^*_{nd}(g, \boldsymbol{\vartheta}) = S_{nd}(G, \boldsymbol{\vartheta}), \tag{2.6}$$

$$\|T_{nd}\| = \|T^*_{nd}\| \geq \frac{1}{2}\|S^*_{nd}\| = \frac{1}{2}\|S_{nd}\|. \tag{2.7}$$

A simple consequence is

Theorem 2.4. *If P_{nd} is a projection of $C(I^d)$ onto Π_{nd} then*

$$\|P_{nd}\| \geq \frac{1}{2}\|S_{nd}\|. \tag{2.8}$$

Above, P_{nd} is a projection of $C(I^d)$ (:= the set of continuous functions of d-variables on $I^d = [-1, 1]^d$) onto Π_{nd} iff it is linear, $P_{nd}(p) = p$ if $p \in \Pi_{nd}$ and $P_{nd}(f) \in \Pi_{nd}$ for any $f \in C(I^d)$.

As an application of Theorem 2.4, see the papers [8–10] dealing with the two-dimensional Lagrange interpolation defined on special node systems.

Remark. Using our methods we intend to settle the "critical cases" in the paper [11, p. 290.] in our forthcoming work.

3. Proofs

3.1. Proof of Theorem 2.1

3.1.1

The relation

$$\|D_{nd}\|_1 = \|S_{nd}\| \tag{3.1}$$

is a simple consequence of the Riesz representation theorem (see [12, IV. 6.3] or [13]).

3.1.2

Next, we prove the relation

$$\|D_{nd}\|_1 \sim (\log n)^d \quad (n \geq 2).$$

Our basic tool is the relation proved by Xu (cf. [5, Lemma 1])

$$D_{nd}(\boldsymbol{\vartheta}) = (-1)^{\left[\frac{d-1}{2}\right]} \sum_{l=1}^{d} \frac{2\cos\frac{\vartheta_l}{2}(\sin\vartheta_l)^{d-2}\mathrm{soc}\frac{2n+1}{2}\vartheta_l}{\prod\limits_{\substack{j=1\\ j\neq l}}^{d}(\cos\vartheta_l - \cos\vartheta_j)}, \tag{3.2}$$

where the function soc (sin or cos) is defined by

$$\mathrm{soc}\vartheta := \begin{cases} \sin\vartheta & \text{if } d \text{ is odd,} \\ \cos\vartheta & \text{if } d \text{ is even.} \end{cases} \tag{3.3}$$

3.1.3

First we prove that $\|D_{nd}\|_1 \le c(\log n)^d$. Indeed, let

$$e_d := \left\{\boldsymbol{\vartheta} : |\vartheta_k| \le n^{-d}, |\vartheta_k - \vartheta_l| \le n^{-d},\ 1 \le k \neq l \le d\right\}. \tag{3.4}$$

Obviously $|e_d| \le A(d)n^{-d}$, where $A(d) > 0$, fixed. If $E_d := \mathbb{T}^d \setminus e_d$ we get

$$\int_{\mathbb{T}^d} |D_{nd}| = \int_{e_d} |D_{nd}| + \int_{E_d} |D_{nd}|. \tag{3.5}$$

By (2.1), $|D_{nd}(\boldsymbol{\vartheta})| \le 2^d \binom{2n+1}{d} \le B(d)n^d$, whence

$$\int_{e_d} |D_{nd}| \le A(d)B(d).$$

At the second integral of (3.5) we estimate only one term on the right-hand side of (3.2) on $F_d := E_d \cap \left[0, \frac{\pi}{2}\right]^d$ and then we use symmetry. By

$$\sin\vartheta_1 \le 2\sin\frac{\vartheta_1 + \vartheta_j}{2} \quad (2 \le j \le d,\ \boldsymbol{\vartheta} \in F_d) \tag{3.6}$$

(consider the cases $\vartheta_1 \le \vartheta_j$ or $\vartheta_1 > \vartheta_j$) and

$$\cos\vartheta_1 - \cos\vartheta_j = 2\sin\frac{\vartheta_1 + \vartheta_j}{2}\sin\frac{\vartheta_1 - \vartheta_j}{2}, \tag{3.7}$$

we get an estimate for s_1 (the first term of (3.2)) as follows:

$$\begin{aligned} |s_1| &\le c \cdot \frac{\prod\limits_{j=2}^{d}\sin\frac{\vartheta_1+\vartheta_j}{2}}{\sin\vartheta_1 \prod\limits_{j=2}^{d}\sin\frac{\vartheta_1+\vartheta_j}{2}\left|\sin\frac{\vartheta_1-\vartheta_j}{2}\right|} \\ &\le \frac{c_1}{\sin\vartheta_1 \prod\limits_{j=2}^{d}\left|\sin\frac{\vartheta_1-\vartheta_j}{2}\right|} \quad (\boldsymbol{\vartheta} \in F_d). \end{aligned}$$

Then, using the substitution $u_1 = \vartheta_1, u_j = \vartheta_j - \vartheta_1$, whence $\vartheta_1 = u_1, \vartheta_j = u_1 + u_j (2 \le j \le d)$, we get

$$\begin{aligned}
\int_{F_d} |s_1| \mathrm{d}\boldsymbol{\vartheta} &\le c_1 \int_{F_d} \frac{1}{\sin \vartheta_1 \prod_{j=2}^{d} \left| \sin \frac{\vartheta_1 - \vartheta_j}{2} \right|} \mathrm{d}\boldsymbol{\vartheta} \\
&\le c_2 V_d \int_{\frac{1}{n^d}}^{\frac{\pi}{2}} \int_{\frac{1}{n^d}}^{\frac{\pi}{2}} \cdots \int_{\frac{1}{n^d}}^{\frac{\pi}{2}} \frac{1}{\prod_{j=1}^{d} u_j} \mathrm{d}u_1 \mathrm{d}u_2 \ldots \mathrm{d}u_d \\
&= c_2 \left(\log \frac{\pi n^d}{2} \right)^d \sim (\log n)^d ,
\end{aligned}$$

where if

$$U_d := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ & & \ldots & & \\ 1 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is the matrix of the transformation, then $V_d = |\det U_d| = 1$.

3.1.4

Here we prove that

$$\|D_{nd}\|_1 \ge c(\log n)^d \quad (n \in \mathbb{N}) \tag{3.8}$$

with a constant $c > 0$ independent of n.

Our proof is based on Fejér's classical example (see [14, Vol. 2, Ch. 2/1]). We suggest the interested reader to examine it carefully.

For every $n \in \mathbb{N}$ we shall construct a trigonometric polynomial $f_n(\mathbf{t}) = \sum_{\mathbf{j}} c_{\mathbf{j}} e^{i\mathbf{j}\cdot\mathbf{t}}$ with

$$\|f_n\| \le 1 \quad \text{and} \quad |S_{nd}(f_n, \mathbf{0})| \ge c(\log n)^d \quad (n \in \mathbb{N}), \tag{3.9}$$

where $c > 0$ is independent of n.

With these polynomials we have

$$\|D_{nd}\|_1 = \|S_{nd}\| = \max_{\substack{g \in C(\mathbb{T}^d) \\ \|g\| \le 1}} \|S_{nd}(g, \cdot)\| \ge |S_{nd}(f_n, \mathbf{0})|$$

which proves (3.8).

For the construction of f_n first we choose real numbers $\alpha_j, \beta_j (j = 1, 2, \ldots, d)$ for which

$$1 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots > \alpha_d > \beta_d > 0. \tag{3.10}$$

Starting from Fejér's classical example $\mathcal{F}_m$ (see (3.12)) let us consider the trigonometric polynomials

$$F_j(t) := \sum_{|k_j|=[n^{\beta_j}]}^{[n^{\alpha_j}]} \frac{1}{k_j} e^{ik_j t} \quad (t \in [0, 2\pi), 1 \le j \le d). \tag{3.11}$$

As we know for the trigonometric polynomials

$$\mathcal{F}_m(t) := \sum_{0<|l|\le m} \frac{1}{l} e^{ilt} \quad (t \in \mathbb{R},\ m \in \mathbb{N}) \tag{3.12}$$

we have

$$|\mathcal{F}_m(t)| = 2\left|\sum_{l=1}^{m} \frac{\sin lt}{l}\right| \le 4\sqrt{\pi} \quad (t \in \mathbb{R},\ m \in \mathbb{N})$$

(see [15,16], [14, Vol. I, (118)]). Therefore we get

$$\begin{aligned} \left|F_j(t)\right| &= \left|\mathcal{F}_{[n^{\alpha_j}]}(t) - \mathcal{F}_{[n^{\beta_j}]-1}(t)\right| \le \left|\mathcal{F}_{[n^{\alpha_j}]}(t)\right| + \left|\mathcal{F}_{[n^{\beta_j}]-1}(t)\right| \\ &\le 8\sqrt{\pi} =: M. \end{aligned} \tag{3.13}$$

Using $\mathbf{e}_j := (0, \ldots, 0, 1, 0, \ldots, 0)(1 \le j \le d)$ (the canonical unit vectors of $\mathbb{R}^d$) we define the polynomial $g_n(\mathbf{t}) =: M^d f_n(\mathbf{t})$ as follows

$$\begin{aligned} g_n(\mathbf{t}) &:= e^{in\mathbf{e}_d\cdot\mathbf{t}} \cdot F_d(-\mathbf{e}_d \cdot \mathbf{t}) \cdot \prod_{j=1}^{d-1} F_j\big((\mathbf{e}_j - \mathbf{e}_d) \cdot \mathbf{t}\big) \\ &= e^{int_d} \sum_{|k_d|=[n^{\beta_d}]}^{[n^{\alpha_d}]} \frac{e^{-ik_d t_d}}{k_d} \prod_{j=1}^{d-1} \left(\sum_{|k_j|=[n^{\beta_j}]}^{[n^{\alpha_j}]} \frac{1}{k_j} e^{i(k_j t_j - k_j t_d)} \right), \end{aligned} \tag{3.14}$$

where $\prod_{j=1}^{0} \cdots := 1$ and $(\mathbf{e}_j - \mathbf{e}_d) \cdot \mathbf{t} = t_j - t_d$.

Using (3.13) we obtain that $|f_n(\mathbf{t})| \le 1 (t \in \mathbb{T}^d)$, i.e. the first requirement of (3.9) holds.

The polynomial $g_n(\mathbf{t})$ can be written as

$$g_n(\mathbf{t}) = \sum\nolimits_{k_1,\ldots,k_d}^{*} \frac{1}{k_1 k_2 \cdots k_d} e^{i\mathbf{k}\cdot\mathbf{t}},$$

where $\mathbf{k} := (k_1, k_2, \ldots, k_{d-1}, n - k_1 - k_2 - \cdots - k_d)$ and the notation $\sum^{*}_{k_1,\ldots,k_d}$ means that we take the summation for indices $[n^{\beta_j}] \le |k_j| \le [n^{\alpha_j}] (1 \le j \le d)$.

The triangular partial sum of the Fourier series of g_n at the point $\mathbf{0}$ is given by

$$S_{nd}(g_n, \mathbf{0}) = \sum_{\substack{k_1,\ldots,k_d \\ |\mathbf{k}|_1 \le n}}^{*} \frac{1}{k_1 k_2 \cdots k_d}. \tag{3.15}$$

Now we prove that in this sum appear *only the positive indices* $k_1, \ldots, k_d$.

First we remark that $n - k_1 - k_2 - \cdots - k_d \geq 0$. Then, by $|k_j| - k_j \geq 0 (1 \leq j \leq d)$, we get

$$\begin{aligned} n \geq |\mathbf{k}|_1 &= |k_1| + |k_2| + \cdots + |k_{d-1}| + |n - k_1 - k_2 - \cdots - k_d| \\ &= \sum_{j=1}^{d-1} \left(|k_j| - k_j\right) + n - k_d \geq n - k_d, \end{aligned}$$

whence we obtain that $k_d \geq 0$. Applying (3.10) we have $k_d > 0$.

Let us suppose that for a fixed index $j^* (1 \leq j^* \leq d-1)$ we have $k_{j^*} < 0$. Since $-k_{j^*} - k_d > 0$ for $n \geq n_0$ (see (3.10)), we would get

$$|\mathbf{k}|_1 = \sum_{j=1}^{d-1} \left(|k_j| - k_j\right) + n - k_d \geq -2k_{j^*} + n - k_d > n - k_{j^*} > n$$

— a contradiction. This means that in (3.15) appear only positive indices $k_1, k_2, \ldots, k_{d-1}$, indeed. Therefore we get (see (3.14))

$$S_{nd}\left(f_n, \mathbf{0}\right) = \frac{1}{M^d} \prod_{j=1}^{d} \left(\sum_{k_j = [n^{\beta_j}]}^{[n_j^{\alpha}]} \frac{1}{k_j} \right) \geq c(\log n)^d$$

which proves the second requirement of (3.9).

Thus the proof of (3.8) is complete. $\square$

3.2. *Proof of* Theorem 2.2.

It is analogous to the classical argument (cf. [7, p. 282]).

3.2.1

First we verify that the integrand in (2.4) is a continuous function of $(\boldsymbol{\vartheta}, \mathbf{t}) \in \mathbb{T}^d \times \mathbb{T}^d$. Indeed, if $\boldsymbol{\vartheta}_0, \mathbf{t}_0$ are fixed then

$$\begin{aligned} |T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta}) - T_{nd}(g_{\mathbf{t}_0}, \boldsymbol{\vartheta}_0)| &\leq |T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta}) - T_{nd}(g_{\mathbf{t}_0}, \boldsymbol{\vartheta})| \\ &\quad + |T_{nd}(g_{\mathbf{t}_0}, \boldsymbol{\vartheta}) - T_{nd}(g_{\mathbf{t}_0}, \boldsymbol{\vartheta}_0)| =: \delta_1 + \delta_2. \end{aligned}$$

Above, $\delta_1 \leq \|T_{nd}\| \|g_{\mathbf{t}} - g_{\mathbf{t}_0}\| < \varepsilon$ if $\|\mathbf{t} - \mathbf{t}_0\|$ is small; on the other hand if $\|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0\|$ is small then $\delta_2 < \varepsilon$, too.

So the integral (2.4) is well defined. Denoting it by $A(g, \boldsymbol{\vartheta})$ we prove that $A(g, \boldsymbol{\vartheta}) = S_{nd}(g, \boldsymbol{\vartheta})$ for any $g \in C(\mathbb{T}^d)$.

First, we verify that $A(g)$ is linear trigonometric projection operator. It is enough to prove the boundedness of $A(g)$. But it follows by

$$\begin{aligned} |A(g, \boldsymbol{\vartheta})| &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t})| \mathrm{d}\mathbf{t} \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|T_{nd}\| \|g_{\mathbf{t}}\| \mathrm{d}\mathbf{t} = \|T_{nd}\| \|g_{\mathbf{t}}\|, \end{aligned}$$

whence $\|A\| \leq \|T_{nd}\|$.

3.2.2

Moreover, we verify that

(a) for $p \in \mathcal{T}_{nd}$, $A(p) = S_{nd}(p)$. Indeed, by $p_{\mathbf{t}}(\boldsymbol{\vartheta}) \in \mathcal{T}_{nd}$ we have that $T_{nd}(p_{\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t}) = p_{\mathbf{t}}(\boldsymbol{\vartheta} - \mathbf{t}) = p(\boldsymbol{\vartheta})$, whence by (2.4), $A(p) = p$, which by $S_{nd}(p) = p$ (cf. [7, p.209]), gives $A(p) = S_{nd}(p)$ if $p \in \mathcal{T}_{nd}$.

(b) Now let $p(\boldsymbol{\vartheta}) = \cos(\mathbf{m} \cdot \boldsymbol{\vartheta})$, $|\mathbf{m}|_1 > n$. We prove that $A(p, \boldsymbol{\vartheta}) = \mathbf{0}$. Indeed, by

$$p_{\mathbf{t}}(\boldsymbol{\vartheta}) = \cos(\mathbf{m} \cdot \boldsymbol{\vartheta}) \cos(\mathbf{m} \cdot \mathbf{t}) - \sin(\mathbf{m} \cdot \boldsymbol{\vartheta}) \sin(\mathbf{m} \cdot \mathbf{t})$$

(to verify, see the exponential form, say) we get

$$\begin{aligned}(2\pi)^d A(p, \boldsymbol{\vartheta}) &= \int_{\mathbb{T}^d} T_{nd}(p, \boldsymbol{\vartheta} - \mathbf{t}) \cos(\mathbf{m} \cdot \mathbf{t}) \mathrm{d}\mathbf{t} \\ &\quad - \int_{\mathbb{T}^d} T_{nd}(q, \boldsymbol{\vartheta} - \mathbf{t}) \sin(\mathbf{m} \cdot \mathbf{t}) \mathrm{d}\mathbf{t} =: I_1 + I_2,\end{aligned}$$

where $q(\boldsymbol{\vartheta}) = \sin(\mathbf{m} \cdot \boldsymbol{\vartheta})$. Consider the first integral, say. By definition, $T_{nd}(p, \boldsymbol{\vartheta} - \mathbf{t})$ is a linear combination of terms having the form $\cos(\mathbf{k} \cdot \mathbf{t})(|\mathbf{k}|_1 \le n)$, whence in I_1 we have integrals of type

$$\int_{\mathbb{T}^d} \cos(\mathbf{k} \cdot \mathbf{t}) \cos(\mathbf{m} \cdot \mathbf{t}) \mathrm{d}\mathbf{t}. \tag{3.16}$$

But they are equal to zero. Indeed, as it is well known (and easy to see) that the functions $\{\mathrm{e}^{i\mathbf{r}\cdot\mathbf{t}}; \mathbf{r} \in \mathbb{Z}_0^d, \mathbf{t} \in \mathbb{T}^d\}$ are mutually orthogonal, i.e.

$$\int_{\mathbb{T}^d} \mathrm{e}^{i\mathbf{r}\cdot\mathbf{t}} \mathrm{e}^{-\mathrm{i}\mathbf{s}\cdot\mathbf{t}} \mathrm{d}\mathbf{t} \ne 0 \quad \text{iff } \mathbf{r} = \mathbf{s};$$

whence by

$$\begin{aligned}2\cos(\mathbf{r} \cdot \boldsymbol{\vartheta}) &= \mathrm{e}^{i\mathbf{r}\cdot\boldsymbol{\vartheta}} + \mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot\boldsymbol{\vartheta}}, \\ 2\mathrm{i}\sin(\mathbf{r} \cdot \boldsymbol{\vartheta}) &= \mathrm{e}^{i\mathbf{r}\cdot\boldsymbol{\vartheta}} - \mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot\boldsymbol{\vartheta}}\end{aligned}$$

(one may verify them by induction for d), the functions

$$\{\mathbf{e}, \cos(\mathbf{r} \cdot \mathbf{t}), \sin(\mathbf{s} \cdot \mathbf{t})\};$$

$\mathbf{r}, \mathbf{s} \in \mathbb{N}^d$, $\mathbf{e} = \{1\}^d$ are mutually orthogonal, too. That means, in (3.16) using $|\mathbf{k}|_1 < |\mathbf{m}|_1$ we get $\mathbf{k} \ne \mathbf{m}$, i.e. the integral is zero, indeed.

The other cases are similar. Using that $S_{nd}(p) = S_{nd}(q) = 0(|\mathbf{m}|_1 > n)$, we get $A(p) = A(q) = S_{nd}(p) = S_{nd}(q) = 0$ at case (b).

3.2.3

Summarizing, we obtained that $S_{nd}(g) = A(g)$ for all trigonometric polynomials, whatever their degree. Using that they form a dense set in $C(\mathbb{T}^d)$ (cf. [2, Theorem 1.7, p. 248]), the operators A and S_{nd} coincide. □

3.2.4

Now (2.5) is immediate: By (2.4) we write

$$|S_{nd}(g, \boldsymbol{\vartheta})| \le \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t})| \mathrm{d}\mathbf{t} \le \|T_{nd}\| \, \|g\|. \quad \square$$

3.3. Proof of Theorem 2.3.

It is very similar to the previous one. For the sake of variety we use the system $\{e^{i\mathbf{r}\cdot\mathbf{t}}\}$.

Let $g(\boldsymbol{\vartheta}) = e^{i\mathbf{k}\cdot\boldsymbol{\vartheta}}$, $|\mathbf{k}|_1 \le n$. Then by $g_{\pm\mathbf{t}}(\boldsymbol{\vartheta}) = e^{i\mathbf{k}\cdot(\boldsymbol{\vartheta}\pm\mathbf{t})}$, whence

$$T^*_{nd}\left(g_{\mathbf{t}} + g_{-\mathbf{t}},\ \boldsymbol{\vartheta} - \mathbf{t}\right) = 2\cos\left(\mathbf{k}\cdot(\boldsymbol{\vartheta} - \mathbf{t})\right)\,\cos(\mathbf{k}\cdot\mathbf{t}).$$

So the integrand in (2.6) is equal to $\cos(\mathbf{k}\cdot\boldsymbol{\vartheta})$ (simple calculation) which is also the right-hand side of (2.6).

(b) If $g(\boldsymbol{\vartheta}) = e^{i\mathbf{m}\cdot\boldsymbol{\vartheta}}$, $|\mathbf{m}|_1 > n$, then the integrand is $T(\boldsymbol{\vartheta} - \mathbf{t})\cos(\mathbf{m}\cdot\mathbf{t})$, where $T \in \mathcal{T}_{nd}$, i.e. the left-hand side is zero, which holds for the right-hand side, too.

Using the above facts, we can get (2.6) as before (cf. 3.2.1–3.2.3).

Finally, using the obvious fact $\|T_{nd}\| = \|T^*_{nd}\|$, (2.7) easily comes from (2.6) (see 3.2.4). □

3.4. Proof of Theorem 2.4.

The map

$$V : f(\mathbf{x}) \to f(\cos\boldsymbol{\vartheta}) \equiv g(\boldsymbol{\vartheta})$$

is an isometric operator with $\|V\| = 1$. If we define the projection in Theorem 2.3 by

$$T_{nd} = V P_{nd} V^{-1},$$

then by (2.7)

$$\frac{1}{2}\|S_{nd}\| \le \|T^*_{nd}\| = \|T_{nd}\| = \|V P_{nd} V^{-1}\|,$$

as it was stated. □

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